MATH GU 4156 **Advanced Probability Theory** Columbia University (Fall 2023)

Lecture 9: Square-Integrable Martingales

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We call a martingale $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}_0}$ **square-integrable** if $\mathbb{E}M_n^2 < \infty$ holds for all n. Then $\mathcal{M}^2$ is a submartingale, with Doob decomposition

$$\mathcal{M}^2 - \langle \mathcal{M} \rangle = \textbf{martingale}$$

for the predictable, increasing sequence,

$$\langle M \rangle_0 = 0;\ \langle M \rangle_n = \sum_{k=1}^{n} \left[\mathbb{E}(M_k^2 | \mathcal{F}_{k-1}) - M_{k-1}^2 \right] = \sum_{k=1}^{n} \mathbb{E}\left[(M_k - M_{k-1})^2 | \mathcal{F}_{k-1} \right], n \in \mathbb{N}_0.$$

Remark 1 *It is instructive to note that $\mathcal{M}^2 - A$ is also a martingale, where*

$$A_0 = 0;\ A_m = \sum_{k-1}^{n} (M_k - M_{k-1})^2 =: [M]_n, n \in \mathbb{N}_0$$

is increasing, adapted, though NOT predictable.

This shows that the requirement of predictability in the Doob decomposition, is essential.

We make now the elementary observations

$$\mathbb{E}[(M_k - M_j)^2 \mathcal{F}_j] = \mathbb{E}(M_k^2 | \mathcal{F}_j) - M_j^2 = \mathbb{E}[\langle M \rangle_k | \mathcal{F}_j] - \langle M \rangle_j$$

$$\mathbb{E}[(M_k - M_j) \cdot \xi] = \mathbb{E}[\xi \cdot (\mathbb{E}(M_k | \mathcal{F}_j) - M_j)] = 0$$

for $k > j$, $\xi \in \mathbb{L}^2(\mathcal{F}_j)$. In particular, **the increments of $\{M_n\}_{n \in \mathbb{N}_0}$ over non-overlapping intervals are orthogonal**, and we have the Pythagorean relationship

$$\mathbb{E}[(M_{n+j} - M_n)^2] = \mathbb{E}[M_{n+j}^2 - M_n^2] = \mathbb{E}[\langle M \rangle_{n+j} - \langle M \rangle_n] = \sum_{k=n+1}^{n+j} \mathbb{E}(M_k - M_{k-1})^2, n \in \mathbb{N}_0.$$

Similarly, if $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}_0}$, $\mathcal{N} = \{N_n\}_{n \in \mathbb{N}_0}$ are square-integrable martingale, the difference

$$\mathcal{M}\mathcal{N} - \langle \mathcal{M}, \mathcal{N} \rangle$$

is a martingale, where

$$\langle \mathcal{M}, \mathcal{N} \rangle := \sum_{k-1}^{n} \mathbb{E}\left[(M+k - M_{k-1})(N_k - N_{k-1}) | \mathcal{F}_{k-1} \right], \; n \in \mathbb{N}_0$$

and $\langle \mathcal{M}, \mathcal{N} \rangle_0 := 0$, the "angle-bracket" of $\mathcal{M}, \mathcal{N}$, is predictable, and satisfies

$$\mathbb{E}\left[(M_{n+j} - M_n)(N_{n+j} - N_n) | \mathcal{F}_n \right] = \mathbb{E}(M_{n+j} N_{n+j} | \mathcal{F}_n) - M_n N_n = \mathbb{E}(\langle \mathcal{M}, \mathcal{N} \rangle_{n+j} | \mathcal{F}_n) - \langle \mathcal{M}, \mathcal{N} \rangle_n.$$

Definition 9.1 *We say that $\mathcal{M}, \mathcal{N}$ are **orthogonal**, if $\langle \mathcal{M}, \mathcal{N} \rangle \equiv 0$; equivalently, if the product $\mathcal{M}\mathcal{N}$ is a martingale.*

Proposition 9.2 *For a square-integrable martingale $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}_0}$ we have*

$$\sup_{n \in \mathbb{N}_0} \mathbb{E}(M_n^2) < \infty \Leftrightarrow \sum_{k \in \mathbb{N}} \mathbb{E}(M_k - M_{k-1})^2 < \infty \Leftrightarrow \mathbb{E}(\langle M \rangle_\infty) < \infty.$$

And in this case, for some $M_\infty \in \mathbb{L}^2$, we have

$$\lim_n M_n = \mathcal{M}_\infty \text{ both a.e. and in } \mathbb{L}^2.$$

Proof: The first claim is clear from the Pythagorean relationship, which gives

$$\sup_{n \in \mathbb{N}_0} \mathbb{E}(M_n^2) - \mathbb{E}(M_0^2) = \sum_{k \in \mathbb{N}} \mathbb{E}(M_k - M_{k-1})^2 = \mathbb{E}(\langle M \rangle_\infty).$$

If these quantities are finite, the sequence $\mathcal{M}$ is bounded in $\mathbb{L}^2$ (thus also uniformly integrable), so the Doob Martingale Convergence Theorem shows that

$$\lim_{n \to \infty} M_n = M_\infty \text{ exists both a.e. and in } \mathbb{L}^1.$$

But then from Fatou and the Pythagorean relationship

$$\begin{aligned}
\mathbb{E}(M_\infty - M_n)^2 &= \mathbb{E}[\lim_{j \to \infty} (M_{n+j} - M_n)^2] \le \liminf_{j \to \infty} \mathbb{E}(M_{n+j} - M_n)^2 \\
&= \lim_{j \to \infty} \sum_{k=n+1}^{n+j} \mathbb{E}(M_k - M_{k-1})^2 = \sum_{k \ge n+1} \mathbb{E}(M_k - M_{k-1})^2 \\
&= \mathbb{E}(\langle M \rangle_\infty - \langle M \rangle_n) < \infty.
\end{aligned}$$

It develops that $\mathbb{E}(M_\infty^2) < \infty$, $M_n \xrightarrow{\mathbb{L}^2} M_\infty$. ■

It turns out that the convergence of a square-integrable martingale $\mathcal{M}$, is tied to the finiteness of the limit

$$\langle M \rangle_\infty = \lim_{t\to\infty} \langle M \rangle_n$$

of its bracket.

Theorem 9.3 (Convergence of Square-Integarble Martingales) *For a square-integrable martingale $\mathcal{M}$, the limit*

$$\lim_{n\to\infty} M_n \textit{ exists in } \mathbb{R} \textit{ a.e. on } \{\langle M \rangle_\infty < \infty\}.$$

And if the increments of this martingale are bounded by some real constant, for $(n, \omega) \in \mathbb{N} \times \Omega$, then

$$\{\lim_{n\to\infty} M_n \textit{ exists in } \mathbb{R}\} = \{\langle M \rangle_\infty < \infty\}, \textit{ mod. } \mathbb{P}.$$

Theorem 9.4 (SLLN for Square-Integarble Martingales) *For a square-integrable martingale $\mathcal{M} = \{M_n\}_{n\in\mathbb{N}_0}$, we have the SLLN*

$$\lim_{n\to\infty} \frac{M_n}{1 + \langle M \rangle_n} = 0, \textit{ a.e. on } \{\langle M \rangle_\infty = \infty\}.$$

Let us go back to some (very) classical Probability Theory.

Example: Series of Independent Random Variables Suppose $\xi_1, \xi_2 ...$ are independent, with $\sigma_k^2 = \mathbb{E}(\xi_k^2) < \infty$ and $\mathbb{E}(\xi_k) = 0$, $\forall k \in \mathbb{N}$. Then

$$M_0 = 0;\ M_n = \sum_{j=1}^{n} \xi_j \ (n \in \mathbb{N})$$

is a square-integrable martingale with $\langle M \rangle_n \sum_{j=1}^{n} \sigma_j^2$; it is also bounded in $\mathbb{L}^2$, if $\sum_{j\in\mathbb{N}} \sigma_j^2 < \infty$.

From the Proposition and Theorem, we deduce the

Kolmogorov Criterion

$$\sum_{k\in\mathbb{N}} \sigma_k^2 < \infty \Longrightarrow \sum_{k\in\mathbb{N}} \xi_k(\omega) \text{ converges in } \mathbb{R} \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

as well as

$$\sum_{k\in\mathbb{N}} \sigma_k^2 < \infty \Longleftarrow \sum_{k\in\mathbb{N}} \xi_k(\omega) \text{ converges in } \mathbb{R} \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

when the $\xi_n(\omega)$ are uniformly bounded, $\forall (n,\omega) \in \mathbb{N} \times \Omega$.

In particular, if $a_1, a_2, ...$ are bounded real number, and $\xi_1, \xi_2, ...$ independent symmetric Bernoullis $\mathbb{P}(\xi_j = \pm 1) = \frac{1}{2}$, we have

$$\sum_{k\in\mathbb{N}} a_k^2 < \infty \Leftrightarrow \sum_{k\in\mathbb{N}} a_k \xi_k(\omega) \text{ converges in } \mathbb{R} \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

A bit more generally, we have the following celebrated result.

Theorem 9.5 (Kolmogorov Three-Series Theorem) *For independent $\xi_1, \xi_2, ...$ the series $\sum_{n\in\mathbb{N}_0} \xi_n$ converges in $\mathbb{R}$ a.e. if, and only if,*

(i) $\sum_{n\in\mathbb{N}_0} \mathbb{P}(|\xi_n| > K) < \infty$;

(ii) $\sum_{n\in\mathbb{N}_0} \mathbb{E}(\xi_n \mathbb{1}_{|\xi_n| \le K}) < \infty$ *converges in* $\mathbb{R}$;

(iii) $\sum_{n\in\mathbb{N}_0} \mathbb{V}ar(\xi_n \mathbb{1}_{|\xi_n| \le K}) < \infty$

hold for some (therefore also for all) $K \in (0, \infty)$.